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$2n$ half sides of the regular polygon; that is, it may be represented as a regular substitution group on letters so selected that each letter corresponds to one of these half sides. Hence *the group of movements of the regular polygon contains one and only one operator which transforms a particular half side of the polygon into a given half side.* As this group of movements is dihedral it may be generated by two of its operators of order two.

It is known that subtraction may be regarded geometrically as a reflection of the point representing the subtrahend on the point midway between the minuend and the origin.[*] If we inscribe in a circle a regular polygon of $2m$ sides, $m>1$, the group of movements of this polygon is clearly generated by a rotation through π on the line of symmetry bisecting the first side and a rotation through π on the y-axis. These rotations are equivalent, respectively, to subtraction from π/m and subtraction from π. If we make $m=2$ we have the group, considered above, formed by the operations of getting the complement and the supplement.

In the second example considered above, $m=4$, and, instead of subtracting from π we subtracted from $\frac{1}{2}\pi$, which is evidently always permissible when m is even and greater than 2. As the table of the trigonometric functions may be confined to the interval $\pi/2m$ provided we can deduce the functions of $\pi-A$, and of $\pi/m-A$ from those of A, this indicates how the necessary Table I may be reduced theoretically to an interval which is less than any given interval. Practically, this method should probably not be employed beyond $m=2$.

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[*] *Annals of Mathematics*, Vol. 6 (1905), p. 41.

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A SET OF INDEPENDENT ASSUMPTIONS FOR PROJECTIVE GEOMETRY.[*]

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INTRODUCTION.

The purpose of the present paper is sufficiently indicated in the title. Incidentally an elementary statement is made of well-understood principles in the foundations of mathematics, and the actual working of these principles is exhibited in treating the set of assumptions for projective geometry. Other sets of assumptions for projective geometry have been given by Veblen and Young[†] and by the present writer.[‡] It is believed that the set of assumptions given in the present paper will lead to a development of Projective Geometry which shall be simpler and at the same time form a more natural sequel to Euclidean Geometry than the developments resulting from the other sets just mentioned.

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[*] The substance of this paper was presented to the American Mathematical Society in February, 1911.

[†] Veblen and Young, *Projective Geometry*, Vol. I.

[‡] Lennes, "Duality in Projective Geometry," *Annals of Mathematics*, Vol. 11 (1911).

It is usual to begin any logical discussion such as the present one with a set of definitions of the technical terms to be used. It is obvious, however, that any definition of such a term can only be made in terms of other technical terms which are equally in need of definition. The definitions of these terms will introduce still other terms which in turn require definition, and so on. Hence the process of definition never terminates. One alternative would be defining in a circle which is not permitted in a logical science. The other alternative is the one in use among mathematicians of the present day, namely, starting with a set of undefined terms* which are usually specified at the outset. In this paper the undefined terms are "point," "line," "plane," and an undefined relation "on."

When certain terms have been designated as undefined other terms may be defined in terms of these. Thus, while point, line, and plane are undefined, a *flat pencil of lines* is defined as "the set of all lines in a plane passing through one point of that plane." The objection may now be urged that this gives a definition of flat pencil in terms of undefined words and hence is no definition at all. The answer is that it is the only kind of definition possible.

However the undefined terms are given a sort of indirect definition by the fundamental assumptions in which they occur.† Indeed the undefined terms have no logical content whatever except as it is given them by the assumptions in which they occur.

Clearly assumptions‡ are necessary in any purely deductive system, for a proof in such a system consists simply in showing that according to certain admitted rules of logic a given proposition is a necessary consequence of certain other propositions. Hence it is as impossible to prove every proposition as it is to define every term.

The set of assumptions for a mathematical science must fulfill certain well established requirements. That is, they must be sufficient, independent and consistent. The most important of these is the first, namely, that all the remaining propositions of the science must follow from the assumptions by purely *formal* deduction. This means that at no stage in the argumentation is the intuition or any consideration whatever aside from the propositions used in making the proof permitted to validate a statement or an argument.

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* It should be clearly understood that there is no such thing as an *undefinable* term. The fact is simply that not *all* terms in any particular logical science can be defined. However, two different writers developing the same subject may use entirely different sets of undefined terms.

† The classical name for what in this paper is called assumption was axiom, which was defined as a self-evident truth. It is obvious, however, that no proposition dealing with abstract, undefined symbols can possibly be self-evident. To avoid the connotation of the old term axiom many recent writers have used other terms such as postulate or assumption. The old Greek distinction between postulate and axiom has entirely disappeared from mathematics (except in some elementary texts whose authors are evidently not acquainted in a vital sense with the modern point of view). This distinction seems to have its basis in a misunderstanding of the nature of unproved propositions in mathematics.

‡ The French term for assumption, which translated reads "unproved proposition," seems the most fortunate of all the names that have been used. An axiom, assumption, postulate, or whatever we call it, is distinguished from the other propositions of the system in which they occur solely by the fact they are accepted arbitrarily and without proof. As a matter of fact there are modern sets of "assumptions" which contain propositions much less obvious to the intuition than are some of the theorems derived from them.

The assumptions must thus contain by implication every theorem in the mathematical science in which they occur. This requirement is by no means satisfied by the usual axioms of Euclidean geometry. Thus it is not possible to prove by means of these axioms that the diagonals of a parallelogram meet in a point.

The second requirement, namely, that the assumptions shall be independent, means that no one of them shall be a logical consequence of all the others. The set given in this paper are proved independent in § 4. A proof that one proposition cannot be proved from a given set of propositions is of comparatively recent origin. The well known proof of Lobachevsky (1829) that the assumption of parallels is not a logical consequence of the remaining assumptions of Euclidean geometry is apparently the first one of this kind. Later completely independent systems of assumptions have been given by Hilbert, Veblen, Huntington, and others. The method of proving the independence of a given set of assumptions is discussed in § 4.

A set of assumptions is consistent if no two contradictory propositions can be deduced from them. The method of proving a set of assumptions consistent is discussed in § 3.

A set of assumptions may or may not possess the further quality of being categorical. This property of a set of assumptions is somewhat more difficult to understand than those just described. The aseumptions of the present paper are not categorical. It is shown in § 3 that all these assumptions are satisfied by certain finite sets of points, lines and planes. Obviously they are also satisfied by the ordinary projective geometry. That is, these assumptions are satisfied by two essentially different systems. If, on the other hand, a set of assumptions is such shat any two systems which satisfy all its assumptions are essentially alike, the set is said to be categorical.*

In the present paper it will turn out that the undefined terms point, line, and plane are used in the sense ascribed to them in ordinary projective geometry. A point is "on" a line or plane if it lies in the line or plane in the ordinary sense. In this case the line and plane are also said to be "on" the point. A line is "on" a plane if it lies in the plane, and in this case the plane is also said to be "on" the line. The statements just given are by no means definitions in the logical sense, but merely informal descriptive statements to aid the intuition in giving concrete form to the abstract symbols and propositions that follow. It should be noted that such concrete form is no part of the logical proceedure but simply an aid to the memory in retaining the various steps.

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* For a more precise statement of this matter see E. V. Huntington, *Annals of Mathematics,* Vol. 6 (1906), pp. 1-43.

One of the most far-reaching theorems of Projective Geometry is the so-called Principle of Duality, namely, *Any valid proposition stated in terms of "point," "line," and "plane" and the relation "on" is valid if point and plane are interchanged.*

In the older treatments of projective geometry this general proposition was not proved, the writers contenting themselves with writing the dual theorems and their proofs side by side, thus exhibiting the principle as each theorem arose. In the treatment of Veblen and Young noted above the theorem is proved in its entirety but only after a considerably difficult and complicated argumentation. The undefined terms of their system are "point," and "line," and the relation "on." "Plane" is then defined as a class of points by means of certain collinearities. Since point is an undefined term, while plane is defined, a completely dual treatment from the outset is impossible.

In the present paper the fundamental propositions form a dual set. That is, they are not changed if the words point and plane are interchanged throughout. Hence it follows without further argument that the set of all theorems provable from these them must be dual. That is, for every such theorem containing point and plane there is another exactly like it except that the words point and plane have changed places.

§ 1. FUNDAMENTAL PROPOSITIONS.

I. *The relation "on" is a reciprocal relation.*

This proposition may be stated more in detail as follows:

1. "The point A is on the line l" is equivalent to "the line l is on the point A."

2. "The point A is on the plane α" is equivalent to "the plane α is on the point A."

3. "The line l is on the plane α" is equivalent to "the plane α is on the line l."

Each of these propositions may be split into two. Thus I may be written:

(a) "The point A is on the line l" implies "the line l is on the point A."

(b) "The line l is on the point A" implies "the point A is on the line l."

The three propositions (1), (2), (3) will be referred to as I_1, I_2, I_3.

II. *If the point A and the plane α are each on a line l then A is on α.*

III_1. *Any two points are on at least one line.*

III_2. *Any two planes are on at least one line.*

IV_1. *Two points are on not more than one line.*

IV_2. *Two planes are on not more than one line.*

DEFINITIONS. Points which are on the same line are *collinear*. Otherwise they are *noncollinear*. Planes which are on the same line are *collinear*, and otherwise they are *noncollinear*.

V_1. *Three points are on at least one plane.*

V_2. *Three planes are on at least one point.*

VI_1. *Three noncollinear points are on not more than one plane.*

VI_2. *Three noncollinear planes are on not more than one point.*

VII. *The class of elements known as lines consists of at least one element.*

$VIII_1$. *On every line there are at least three points.*

$VIII_2$. *On every line there are at least three planes.*

IX_1. *On every plane there are at least three noncollinear points.*

IX_2. *On every point there are at least three noncollinear planes.*

Propositions I, ..., IX_2 are obviously dual with respect to point and plane.

§ 2. ASSUMPTIONS AND THEOREMS.

$I_{1,2,3}$; II, III_2, V_1, $V_{1,2}$; VI_1, VII; $VIII_{1,2}$; IX_1 form a set of independent assumptions and all theorems of this section are based on them. III_1; IV_2; VI_2; IX_2 are theorems. For proofs of the independence of the postulates see § 4. III_1, IV_2, VI_2 and IX_2 are proved in this section.

IV_2, *Two planes are on not more than one line.*

PROOF. Given the distinct planes α and β which we suppose to be on each of the two lines l_1 and l_2. By $VIII_1$ there are three points on each line, and hence by IV_1 there are among these points a set of three noncollinear points A, B, C. By II each of the points A, B, C is on α and also on β. But by VI_1 A, B, C are on only one plane.

1_1. THEOREM. *If a point is on each of two planes then the point is on the line which is on the two planes.*

PROOF. Two planes α and β are on one and only one line l (III_2, IV_2); on this line are at least two points A and B. If now there is a point C on both α and β but not on l then A, B, C are noncollinear and hence by VI_1 α and β are the same plane.

VI_2. *Three noncollinear planes are on not more than one point.*

PROOF. If there are two points A and B on each of the planes α, β, γ and if α, β are on the line l_1 and β and γ are on the line l_2, then A and B are on l_1 and also on l_2, and hence l_1 and l_2 are identical, and hence α, β, γ are collinear.

2_1. *Not all points on a plane.*

PROOF. Let α be any plane. By VII, $VIII_2$ at least a plane β distinct from α. By IX_1, II not all points on β are on α. Hence not all points on α.

III_1. *Any two points are on a line.*

PROOF. Let A and B be any two points. By VII a line l exists. If A and B are on l then the theorem is verified. If A and B are not on l there is a point C on l (not collinear with A and B). ($VIII_1$, IV_1). Let D be a point not on the plane α of A, B, C. Then A, B, D are on a plane β. α and β are on one and only one line l_1 (III_2, IV_2). But A and B are on both α and β. Hence l_1 is on A and B (1).

IX_2. *On every point there are at least three noncollinear planes.*

PROOF. Let A be the given point. By VII and VIII_1 there exists at least one other point B, and by III_1 there is a line l on A and B. By VIII_2 there are three distinct planes α, β, γ on l, and on each plane at least one point not on l. Let C, D, E be such points, one on each of the three planes. Then A, C, D are on a plane Δ. This plane Δ is not on l for in that case it would be on both A and B (II), and hence would be identical with both α and β. But α and β are both on l, whence by (II) any point on both α and β is on l. Therefore Δ is not collinear with α and β. That is, α, β, Δ are three noncollinear planes on A.

Propositions I, ..., IX_2 are now completely established (either by assumption or by proof) and hence we state without further argument the space dual of (1_1).

1_2. THEOREM. *If a plane is on each of two points then the plane is on the line which is on these two points.*

3_1. THEOREM. *If two lines are on a plane they are on a point.*

PROOF. Denote the given plane by α and the given lines on it by l_1 and l_2. By VIII_2, IX_1 there is a point C not on α. Let A and B be two points on l_1 and DE two points on l_2. Then ABC are noncollinear as are also CDE (2_1). Denote the planes on these points by β and γ, respectively. By V_2 there is a point F on the planes α, β, γ. Thus F is on l_1 and l_2 (1_2).

3_2. THEOREM. *If two lines are on a point they are on a plane.*

PROOF. This is the dual of (3_1).

Of the usual propositions in projective geometry on the incidence of points, lines, and planes those that now remain are obvious and corollaries of the preceding propositions of this section.

Inasmuch as only a finite number of elements are required to satisfy either set of assumptions (see § 3) some theorems may be trivial, since there may not be a sufficient number of elements present to satisfy the hypothesis. Thus if only the elements required by the assumptions are permitted the Theorem of Desargues is trivial since only seven points are on any one plane.

§ 3. CONSISTENCY OF THE ASSUMPTIONS.

We now inquire whether the given set of assumptions is consistent, independent, and categorical.

That a set of assumptions is consistent is proved by exhibiting a concrete system in which the assumptions are satisfied and which for some reason is regarded as self-consistent. The propositions of § 1 are all satisfied by a system given below in which points are the capital letters, lines the columns each containing three letters, and each set of seven lines forming a rectangle, is a plane.*

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* Finite geometries of which the one given here is a type have been studied by a number of writers. Tactical memoranda by E. H. Moore, *American Journal of Mathematics*, Vol. 18 (1896), pp. 264-303, contains a rich collection

(A)

$$(1)\ \begin{array}{ccccccc} A & B & C & D & E & F & G \\ B & C & D & E & F & G & A \\ D & E & F & G & A & B & C \end{array} \quad (2)\ \begin{array}{ccccccc} A & B & H & D & I & J & K \\ B & H & D & I & J & K & A \\ D & I & J & K & A & B & H \end{array} \quad (3)\ \begin{array}{ccccccc} A & B & M & D & L & N & O \\ B & M & D & L & N & O & A \\ D & L & N & O & A & B & M \end{array}$$

$$(4)\ \begin{array}{ccccccc} B & C & K & E & L & M & J \\ C & K & E & L & M & J & B \\ E & L & M & J & B & C & K \end{array} \quad (5)\ \begin{array}{ccccccc} B & C & N & E & H & I & O \\ C & N & E & H & I & O & B \\ E & H & I & O & B & C & N \end{array} \quad (6)\ \begin{array}{ccccccc} C & D & I & F & K & L & O \\ D & I & F & K & L & O & C \\ F & K & L & O & C & D & I \end{array}$$

$$(7)\ \begin{array}{ccccccc} C & D & H & F & J & M & N \\ D & H & F & J & M & N & C \\ F & J & M & N & C & D & H \end{array} \quad (8)\ \begin{array}{ccccccc} D & E & O & G & H & J & L \\ E & O & G & H & J & L & D \\ G & H & J & L & D & E & O \end{array} \quad (9)\ \begin{array}{ccccccc} D & E & M & G & K & I & N \\ E & M & G & K & I & N & D \\ G & K & I & N & D & E & M \end{array}$$

$$(10)\ \begin{array}{ccccccc} E & F & L & A & I & N & J \\ F & L & A & I & N & J & E \\ A & I & N & J & E & F & L \end{array} \quad (11)\ \begin{array}{ccccccc} E & F & K & A & O & H & M \\ F & K & A & O & H & M & E \\ A & O & H & M & E & F & K \end{array} \quad (12)\ \begin{array}{ccccccc} F & G & J & B & O & K & N \\ G & J & B & O & K & N & F \\ B & O & K & N & F & G & J \end{array}$$

$$(13)\ \begin{array}{ccccccc} F & G & H & B & L & I & M \\ G & H & B & L & I & M & E \\ B & L & I & M & F & G & H \end{array} \quad (14)\ \begin{array}{ccccccc} G & A & I & C & J & O & M \\ A & I & C & J & O & M & C \\ C & J & O & M & G & A & I \end{array} \quad (15)\ \begin{array}{ccccccc} G & A & K & C & H & L & N \\ A & K & C & H & L & N & G \\ C & H & L & N & G & A & K \end{array}$$

In this system there are three planes and also three points on every line. Thus the planes (1), (2), (3) are on the line $\begin{matrix} A \\ B \\ D \end{matrix}$ as are also the points A, B, D. There are seven points on each plane and seven planes on every point. That all the assumptions of § 1 are satisfied by this system is easily verified.

§ 4. Independence of the Assumptions.

We now prove that $I_{1,2,3}$; II; III_2, IV_1; $V_{1,2}$; VI_1; VII, $VIII_{1,2}$; IX_1 of § 1 form a set of independent postulates. $I_{1,2,3}$; II; III_2, IV_1; $V_{1,2}$; VI_1 are to be understood as not affirming the *existence* of points, lines and planes. That is, these assumptions would all be satisfied by an entirely vacuous system. For convenience of reference these assumptions are here restated in more explicit form.

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of such configurations though they are not regarded as finite geometries. Moore first used the matricular notation in this connection. Cf. Veblen and Young, *Projective Geometry,* Vol. 1. In this notation the configuration here given is

$$\begin{array}{ccc} 15 & 3 & 7 \\ 3 & 35 & 3 \\ 7 & 3 & 15 \end{array}$$

This particular configuration is studied in detail by Moore in a paper on the General Equation of the Seventh and Eighth Degree, *Mathematische Annalen,* Vol. 51 (1899), pp. 417-444. It is also studied by George M. Conwell, *Annals of Mathematics,* Vol. 11 (1910), second series, pp. 60-76. Veblen and Bussey, *Transactions of the American Mathematical Society,* Vol. 7 (1906). pp. 241-259, treated these configurations explicitely as finite geometries and discovered the set of all such geometries.

I_1. *"A point A is on a line l" is equivalent to "the line l is on the point A."*

I_2. *"A point A is on a plane* α*" is equivalent to "the plane* α *is on the point A."*

I_3. *"A line l is on a plane* α*" is equivalent to "the plane* α *is on the line l."*

II. *If a point A and a plane* α *are both on a line l then A is on* α.

III_2. *If planes and lines exist then any two planes are on at least one line.*

IV_1. *If points and lines exist then any two points are on not more than one line.*

V_1. *If points and planes exist then any three points are on at least one plane.*

V_2. *If planes and points exist then any three planes are on at least one point.*

VI_1. *If points and planes exist then three noncollinear points are on not more than one plane.*

VII. *The class whose elements are lines contains at least one element.*

VIII_1. *On every line there are at least three points.*

VIII_2. *On every line there are at least three planes.*

IX_1. *On every plane there are at least three noncollinear points.*

INDEPENDENCE OF I_1. To prove the independence of I_1 modify the concrete system (A) of § 3 so that the line $\begin{array}{c}A\\B\\D\end{array}$ shall not be *on* the point A while A is *on* this line. In this system every arsumption except I_1 is verified.

I_2. To prove the independence of I_2 modify the system (A) so that plane (1) shall be *on* the points on which (2) now is, and (2) shall be *on* the points on which (1) now is. The points (1) and (2) remain *on* these planes.

I_3. Modify the concrete system (A) so that the line $\begin{array}{c}A\\B\\D\end{array}$ is not on the plane (1) but plane (1) is on the line.

II. Modify the system (A) so that the points which are now on plane (1) shall be on plane (2), and those now on plane (2) shall be on plane (1). Likewise plane (1) shall be on the points now on plane (2), and plane (2) on the points now on plane (1). Points and lines, and planes and lines are related as in system (A). Thus the line $\begin{array}{c}E\\F\\A\end{array}$ remains *on* the plane (1) and the points E, F, A are on this line while they are also *on* the plane (3).

III_2. In system (A) omit the line $\begin{array}{c}A\\B\\D\end{array}$ but not the points A, B, D.

IV_1. In system (A) add the point G to any line on which it now is not.

V_1. To the system (A) add a point Q which shall be on no line and on no plane.

V_2. To prove the independence of this assumption considsr an ordinary projective geometry with one point removed. Then any three planes which pass through this point are *on* no common point.

VI_1. In the system (A) let the point C be *on* plane (2) (as well as on plane (1)) but not on any line in plane (2).

VII. An entirely vacuous system containing no points, lines, or planes.

VIII_1. In the system (A) omit all points and planes.

VIII_2. In the system (A) omit all planes.

IX_1. Consider a system consisting of planes (1), (2), (3) of (A) modified as follows: $\overset{A}{\underset{D}{B}}$ is a line on each plane and A, B, D are points. The remaining points and lines now in these planes shall be some other distinct entities different from either points or lines. Then we have a system consisting of one line with three points on it, and also three planes. Clearly the points are distinct among themselves, as are also the planes.

Hence we have showed that the set of assumptions of this paper are independent.